Implementing a binomial tree

Tutorial 4, Math 3FM3

Notations:

- S:stock price
- X:strike price
- σ :volatility
- r:risk free rate
- τ :maturity time
- M: time steps
- $\Delta_t = \tau/M$: length of each time section

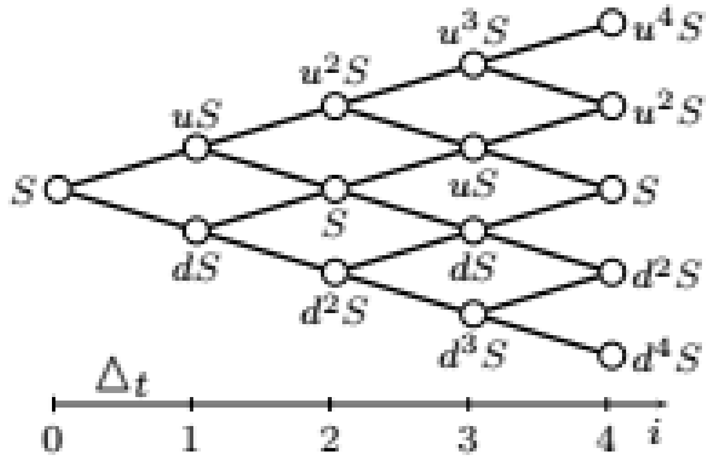
Preliminary

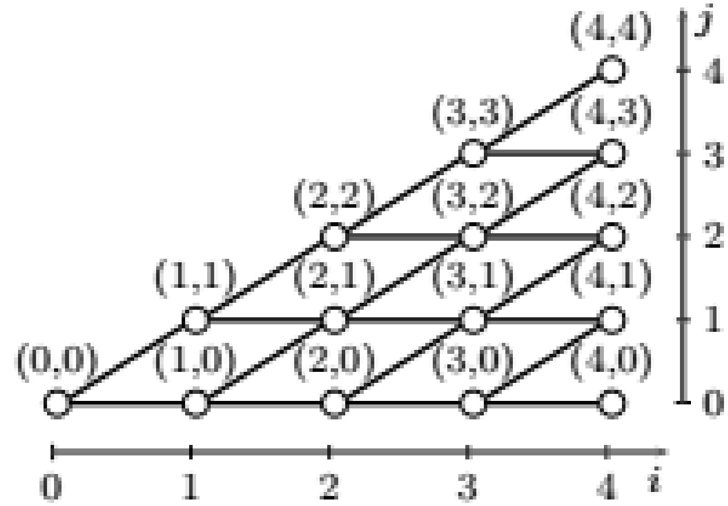
- $E(S_{\Delta t}/S_0)=\tilde{p}\,u+(1-\tilde{p})d=\exp(r\,\Delta_t)$
- we can solve $\tilde{p}=(\exp(r\,\Delta_t)-d)/(u-d)$.
- $Var(S_{\Delta t}/S_0)= Var(S_{\Delta t})/(S_0)^2 =\tilde{p}u^2+(1-\tilde{p})d^2-(\tilde{p}u+(1-\tilde{p})d)^2$
- Also by BSM, $Var(S_{\Delta t})= (S_0)^2 (\exp(2r\,\Delta_t +\sigma^2 \Delta_t)-\exp(2r\,\Delta_t))$
- Cox et al. (1979) suggest the approximative solutions

$$u=\exp(\sigma\sqrt{\Delta_t}),\mathrm{d}=\exp(-\sigma\sqrt{\Delta_t})$$

Growing the tree

- $S_{i,j} = Su^j d^{i-j}$





Algorithm

1: **initialise** $\Delta_t = \tau/M$, $S_{0,0} = S$, $v = e^{-r\Delta_t}$
2: **compute** $u = e^{\sigma\sqrt{\Delta_t}}$, $d = 1/u$, $\tilde{p} = (e^{r\Delta_t} - d)/(u - d)$
3: $S_{M,0} = S_{0,0} d^M$
4: **for** $j = 1 : M$ **do**
5: $\quad S_{M,j} = S_{M,j-1}\, u/d$ # initialise asset prices at maturity
6: **end for**
7: **for** $j = 0 : M$ **do**
8: $\quad C_{M,j} = \max(S_{M,j} - X, 0)$ # initialise option values at maturity
9: **end for**
10: **for** $i = M-1 : -1 : 0$ **do**
11: $\quad$ **for** $j = 0 : i$ **do**
12: $\qquad C_{i,j} = v\,(\tilde{p}\, C_{i,j+1} + (1 - \tilde{p})\, C_{i,j})$ # step back through the tree
13: $\quad$ **end for**
14: **end for**
15: $C_0 = C_{0,0}$

- Replace $\max(S_{M,j} - X, 0)$ in Step 8 by $\max(X - S_{M,j}, 0)$ to get the price of put option.

Numerical Implementation in Matlab and R

- Code attached